MECHANICAL ENGINEERING DEPARTMENT
UNITED STATES NAVAL ACADEMY

EM423 - INTRODUCTION TO MECHANICAL VIBRATIONS

CONTINUOUS SYSTEMS - LONGITUDINAL VIBRATION IN RODS

SYMBOLS

r	Mass density
A_X	Cross-sectional area
u	Longitudinal deflection from equilibrium position
x	Distance along the rod
P	Internal longitudinal force (varies along the rod)

INTRODUCTION

This theory is applicable to longitudinal vibration in slender rods. With this type of vibration, very small amplitudes of motion can produce very large forces. Typical problems include damage to thrust bearings in ship propulsion systems, noise and damage caused by engine support props in helicopters, and radiated noise problems in submarines.

ASSUMPTIONS

1. The rod is thin compared to its length.

2. The rod is uniform, homogeneous and isotropic.

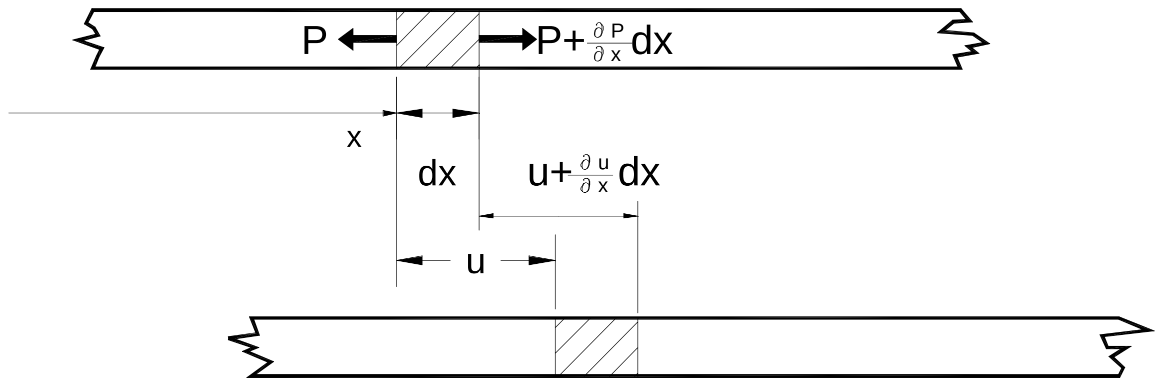
3. The material is within the elastic limit, and obeys Hooke's Law.

4. Plane sections remain plane.

5. The lateral deflection caused by changes in length of the rod is small.

THEORY

Consider a small element of rod:



The element changes length by an amount $\left(\frac{\partial u}{\partial x}\right)dx$ and so:

$$\text{strain} = \frac{(\text{change in length})}{(\text{original length})} = \frac{\partial u}{\partial x}$$

From Hooke's Law, E = (stress) / (strain), and (stress) = (force) / (area), so

$$E = \frac{\left(\frac{P}{A_x}\right)}{\left(\frac{\partial u}{\partial x}\right)} \quad \text{hence} \quad \frac{\partial u}{\partial x} = \frac{P}{A_x E}$$

$$\text{therefore} \quad \frac{\partial P}{\partial x} = A_x E\,\frac{\partial^2 u}{\partial x^2}$$

<u>Resolve forces along the rod.</u>

$$\left(P + \frac{\partial P}{\partial x}d\,x\right) - P = \left(r\,A_x d\,x\right)\frac{\partial^2 u}{\partial t^2}$$

Subsituting for $\frac{\partial P}{\partial x}$ and rearranging yields:

$$\boxed{\begin{gathered}\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \\ \text{with } c = \sqrt{\frac{E}{r}} = \text{ wave velocity}\end{gathered}}$$

This is the same wave equation as derived for string vibrations, but with a different wave velocity. Using the same method to solve the equation, firstly separate the variables.

$$u(x,t) = U(x).G(t)$$

$$\text{with} \quad U(x) = A.\sin\left(\frac{wx}{c}\right) + B.\cos\left(\frac{wx}{c}\right)$$

$$G(t) = \sin(wt)$$

Hence:

$$\boxed{u(x,t) = U(x).G(t) = \left(A.\sin\left(\frac{wx}{c}\right) + B.\cos\left(\frac{wx}{c}\right)\right)\sin(wt)}$$

<u>COMMON BOUNDARY CONDITIONS</u>

<u>For a FREE end</u> the internal stress is zero.

$$\text{stress} = E\frac{\partial u}{\partial x}$$

$$\text{therefore} \quad \frac{\partial u}{\partial x} = 0$$

<u>For a FIXED end</u> the displacement is zero.

$$u = 0$$

NATURAL FREQUENCIES OF A FREE-FREE ROD, length = L

$$u(x,t) = \left(A.\sin\left(\frac{wx}{c}\right) + B.\cos\left(\frac{wx}{c}\right)\right)\sin(wt)$$

$$\text{therefore } \frac{\partial u}{\partial x} = \left(A.\cos\left(\frac{wx}{c}\right) - B.\sin\left(\frac{wx}{c}\right)\right)\left(\frac{w}{c}\right)\sin(wt)$$

$$\text{at } x = 0;\ \frac{\partial u}{\partial x} = 0 \quad \text{hence } A = zero$$

$$\text{also at } x = L;\ \frac{\partial u}{\partial x} = 0$$

This is only true for all time if $\sin\left(\frac{wL}{c}\right) = 0$, so

$$\frac{w_n L}{c} = np \ \ , n = 1, \ 2, \ .....$$

$$\text{from which } w_n = \frac{npc}{L} = \frac{np}{L}\sqrt{\frac{E}{r}}$$

$$\text{and } f_n = \frac{w_n}{2p} = \frac{n}{2L}\sqrt{\frac{E}{r}}$$

NATURAL FREQUENCIES OF A FIXED-FREE ROD, length = L

$$u(x,t) = \left(A.\sin\left(\frac{wx}{c}\right) + B.\cos\left(\frac{wx}{c}\right) \right)\sin(wt)$$

$$\text{therefore } \frac{\partial u}{\partial x} = \left(A.\cos\left(\frac{wx}{c}\right) - B.\cos\left(\frac{wx}{c}\right) \right)\left(\frac{w}{c}\right)\sin(wt)$$

$$\text{at } x = 0;\ u = 0 \quad \text{hence B} = zero$$

$$\text{also at } x = L;\ \frac{\partial u}{\partial x} = 0$$

$$\text{so } \cos\left(\frac{wL}{c}\right) = 0$$

$$\text{and } \frac{w_n L}{c} = \left(n - \frac{1}{2}\right)p \ ,\ n = 1,\ 2,\ .....$$

Rearrange for the natural frequencies:

$$w_n = \frac{\left(n - \frac{1}{2}\right)p\,c}{L} = \frac{\left(n - \frac{1}{2}\right)p}{L}\sqrt{\frac{E}{r}}$$

$$\text{and } f_n = \frac{w_n}{2p} = \frac{\left(n - \frac{1}{2}\right)}{2L}\sqrt{\frac{E}{r}}$$

DISCUSSION

1. The solution for longitudinal vibrations in a rod is similar to flexural vibrations of a taught string, but with a different wave speed.

2. This means the same discussion applies to rod vibrations. The only difference is that, for graphical presentation, longitudinal displacements are usually drawn as if they were transverse.

EXAMPLES

1. A steel propulsion shaft on a ship is 25 m long. Its outside and inside diameters are 32 cm and 15 cm respectively. What are the first 2 natural frequencies of longitudinal vibration? Solve the problem 2 times. First, assume the gearbox and propeller are very light. Second, assume the gearbox is light, but the propeller is very heavy.

$L = 25$ m; $= 7843$ kg/m^3; $E = 210$ kN/mm^2

<u>Free-free</u>

$$f_n = \frac{n}{2L}\sqrt{\frac{E}{r}}$$

$$f_1 = \frac{1}{2\times 25}\sqrt{\frac{210\times 10^9}{7843}} = 103.5 \text{ Hz}$$

$$f_2 = \frac{2}{2\times 25}\sqrt{\frac{210\times 10^9}{7843}} = 207.0 \text{ Hz}$$

<u>Fixed-free</u>

$$f_n = \frac{\left(n - \frac{1}{2}\right)}{2L}\sqrt{\frac{E}{r}}$$

$f_1 = 51.7$ Hz
$f_2 = 155.2$ Hz

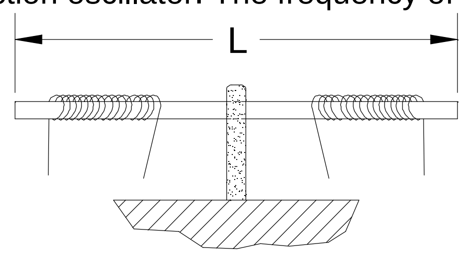
2. What difference does it make to the previous question if you assume both the propeller and gearbox are very heavy?

The solution is identical to that for example 1. Why?

ASSIGNMENTS

1. Find, from first principles, the velocity of longitudinal waves along a thin steel rod.

2. The figure shows a schematic of a magnetostriction oscillator. The frequency of the oscillator is determined by the length of the nickel alloy rod, which generates an alternating voltage in the surrounding coils equal to the frequency of the fundamental longitudinal vibration of the rod. Determine the full length, L, of the rod for a frequency of 20 kHz.



3. (extra credit) A uniform rod of length L and cross-sectional area A is fixed at the upper end and is loaded with a weight W on the other end. Show that the natural frequencies are determined from the equation:

$$wL\sqrt{\frac{r}{E}}\tan\left(wL\sqrt{\frac{r}{E}}\right)=\frac{A_x\, rLg}{W}$$

SOLUTIONS

1. Find, from first principles, the velocity of longitudinal waves along a thin steel rod.

See the Course Handout
CONTINUOUS SYSTEMS – LONGITUDINAL WAVES IN RODS

2. The figure shows a schematic of a magnetostriction oscillator. The frequency of the oscillator is determined by the length of the nickel alloy rod, which generates an alternating voltage in the surrounding coils equal to the frequency of the fundamental longitudinal vibration of the rod. Determine the full length, L, of the rod for a frequency of 20 kHz.

We consider half of the rod, with boundary conditions of fixed at one end ($x = 0$), and free at the other end ($x = L$).

$$u(x,t) = \left\{A.\sin\left(\frac{wx}{c}\right) + B.\cos\left(\frac{wx}{c}\right)\right\}\sin(wt)$$

$$\frac{\partial u}{\partial x} = \left\{A.\cos\left(\frac{wx}{c}\right) - B.\sin\left(\frac{wx}{c}\right)\right\}\left(\frac{w}{c}\right)\sin(wt)$$

$$\text{At } x = 0 \text{ ; } u = 0 \text{ hence } B = zero$$

$$\text{At } x = L \text{ ; } \frac{\partial u}{\partial x} = 0$$

$$\text{This is only true for at time if } \cos\left(\frac{wL}{2c}\right) = 0$$

$$\text{so } \left(\frac{w_n L}{2c}\right) = \left(n - \frac{1}{2}\right)p \quad n = 1,\ 2,\ 3,\ \ldots\ldots$$

$$\text{from which } w_n = 2p f_n = \frac{(2n-1)pc}{L} = \frac{(2n-1)p}{L}\sqrt{\frac{E}{r}}$$

$$\text{and } L = \frac{(2n-1)}{2f_n}\sqrt{\frac{E}{r}}$$

$$\text{hence} \quad \frac{(2-1)}{2\times 20{,}000}\sqrt{\frac{207\times 10^{9}}{8580}} = 0.1228 \text{ m} = 122.8 \text{ mm}$$

3. (extra credit) A uniform rod of length L and cross-sectional area A is fixed at the upper end and is loaded with a weight W on the other end. Show that the natural frequencies are determined from the equation:

$$wL\sqrt{\frac{r}{E}}\tan\left(wL\sqrt{\frac{r}{E}}\right) = \frac{A_x rLg}{W}$$

The solution to extra credit problems is available from your instructor.